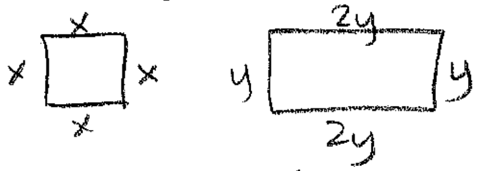
A piece of wire 34 inches long is cut into two pieces. One piece is bent into a square and the other piece is bent into a rectangle whose width is twice its length. Each side of the square and of the rectangle must be at least 1 inch long. **SCORE: _____ / 30 PTS**

<u>You must use calculus to solve the following problems.</u>

[a] What is the largest total area that can be enclosed in the two shapes ?



MAXIMIZE $A = x^2 + 2y^2$

$4x + 6y = 34$

$2x + 3y = 17 \longrightarrow x \geq 1 \Rightarrow y \leq 5$

$x = \frac{1}{2}(17 - 3y)$

$A = \frac{1}{4}(17-3y)^2 + 2y^2$ ON $y \in [1, 5]$

$A' = \frac{1}{2}(17-3y)(-3) + 4y$ IS DEFINED ON $[1,5]$

$= \frac{9}{2}y - \frac{51}{2} + 4y = 0$

$9y - 51 + 8y = 0$

$17y = 51$

$y = 3 \longrightarrow x = 4$

y	A
1	$\frac{1}{4}(14^2) + 2(1^2) = 49 + 2 = 51$
3	$\frac{1}{4}(8^2) + 2(3^2) = 16 + 18 = 34$
5	$\frac{1}{4}(2^2) + 2(5^2) = 1 + 50 = 51$

THE LARGEST AREA IS 51 SQUARE INCHES

[b] Find the dimensions of the square and the rectangle which give the largest total area.

EITHER 7" × 7" AND 1" × 2"

OR 1" × 1" AND 5" × 10"

Find $\int \frac{(3x^3-2)^2}{5x^4}\,dx$. SCORE: _____ / 15 PTS

$$= \int \frac{9x^6 - 12x^3 + 4}{5x^4}\,dx$$

$$= \int \left(\frac{9}{5}x^2 - \frac{12}{5}x^{-1} + \frac{4}{5}x^{-4}\right)dx$$

$$= \frac{9}{5}\frac{1}{3}x^3 - \frac{12}{5}\ln|x| + \frac{4}{5}\left(-\frac{1}{3}\right)x^{-3} + C$$

$$= \frac{3}{5}x^3 - \frac{12}{5}\ln|x| - \frac{4}{15}x^{-3} + C$$

Find $\lim_{x\to 0^+} \sqrt{x}\ln x$. $0 \cdot -\infty$ SCORE: _____ / 15 PTS

$$= \lim_{x\to 0^+} \frac{\ln x}{x^{-\frac{1}{2}}} \qquad \frac{-\infty}{\infty}$$

$$= \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{2}x^{-\frac{3}{2}}}$$

$$= \lim_{x\to 0^+} -2x^{\frac{1}{2}} = 0$$

Does Rolle's Theorem apply to the function $f(x)=\sqrt[3]{x^2-4x+3}$ on the interval $[-1, 5]$? SCORE: _____ / 15 PTS

(That is, are all conditions of Rolle's Theorem true for $f(x)=\sqrt[3]{x^2-4x+3}$ on the interval $[-1, 5]$?)
If yes, find the value of c guaranteed by Rolle's Theorem. If no, explain why not.

$$f'(x) = \frac{1}{3}(x^2-4x+3)^{-\frac{2}{3}}(2x-4)$$

IS UNDEFINED @ $x^2-4x+3=0$

$(x-1)(x-3)=0$

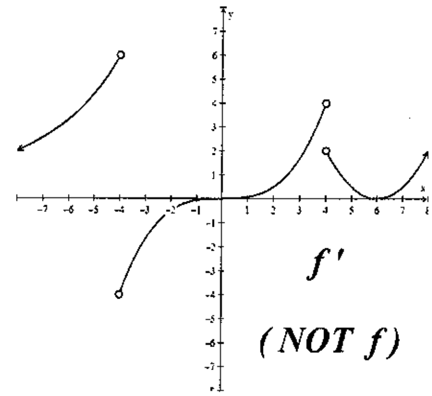
$x = 1, 3 \in [-1,5]$

f IS NOT DIFFERENTIABLE ON $[-1,5]$

SO ROLLE'S THEOREM DOES NOT APPLY

$f(x)$ is a continuous function whose derivative $f'(x)$ is shown on the right. **SCORE: _____ / 24 PTS**

The following questions are about the function f, **NOT THE FUNCTION f'**



[a] Find the $x-$coordinates of all inflection points of f.
Justify your answer very briefly.

f' CHANGES FROM INCR TO DECR: $x=4$
DECR INCR: $x=6$

[b] Find the intervals over which f is decreasing.
Justify your answer very briefly.

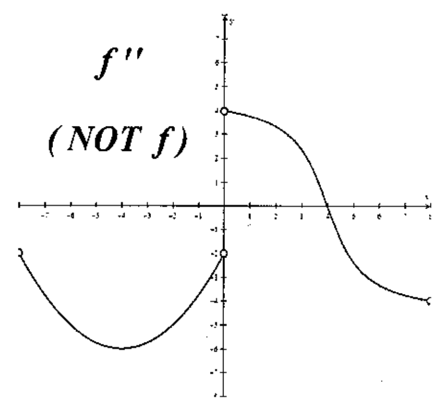
$f'<0$: $(-4,0)$

[c] Find all critical numbers of f, and state what the First Derivative Test tells you about each one.
Justify your answer very briefly.

$f'=0$: 0 f' CHANGES FROM $-$ TO $+$ $\rightarrow$ LOCAL MIN
6 $+$ $+$ $\rightarrow$ NOT EXTREMA
f' DNE : -4 $+$ $-$ $\rightarrow$ LOCAL MAX
4 $+$ $+$ $\rightarrow$ NOT EXTREMA

$f(x)$ is a continuous and differentiable function whose second derivative $f''(x)$ is shown on the right. **SCORE: _____ / 12 PTS**

The following questions are about the function f, **NOT THE FUNCTION f''**



[a] If $f'(6)=0$,
what does the Second Derivative Test tell you about the point $(6, f(6))$?
Justify your answer very briefly.

$f''(6)<0$ $\rightarrow$ LOCAL MAX

[b] Find the intervals over which f is concave up.
Justify your answer very briefly.

$f''>0$: $(0,4)$

Graph $f(x) = \dfrac{x}{(2-x)^3} - 3$ using the process shown in lecture and in the website handout. **SCORE: _____ / 40 PTS**

The first and second derivatives are $f'(x) = (2-x)^{-3} + 3x(2-x)^{-4}$ and $f''(x) = 6(2-x)^{-4} + 12x(2-x)^{-5}$.

Do NOT find x–intercepts.
Complete the table below, after showing relevant work (except for entries marked ★).
You will NOT receive credit for the entries in the table if the relevant work is missing.

★ Domain	★ Discontinuities	y – intercepts **ONLY**	One sided limits at each discontinuity (write using proper limit notation)	
$x \neq 2$	$x = 2$	$(0, -3)$	$\lim_{x\to 2^-} f(x) = \infty$	$\lim_{x\to 2^+} f(x) = -\infty$
Horizontal Asymptotes	**Intervals of Increase**	**Intervals of Decrease**	**Intervals of Upward Concavity**	**Intervals of Downward Concavity**
$y = -3$	$(-1, 2)$, $(2, \infty)$	$(-\infty, -1)$	$(-2, 2)$	$(-\infty, -2)$, $(2, \infty)$
Vertical Tangent Lines	**Horizontal Tangent Lines**	**Local Maxima**	**Local Minima**	**Inflection Points**
NONE	$x = -1$	NONE	$(-1, -3\frac{1}{27})$	$(-2, -3\frac{1}{32})$

$f(0) = \frac{0}{2^3} - 3 = -3$

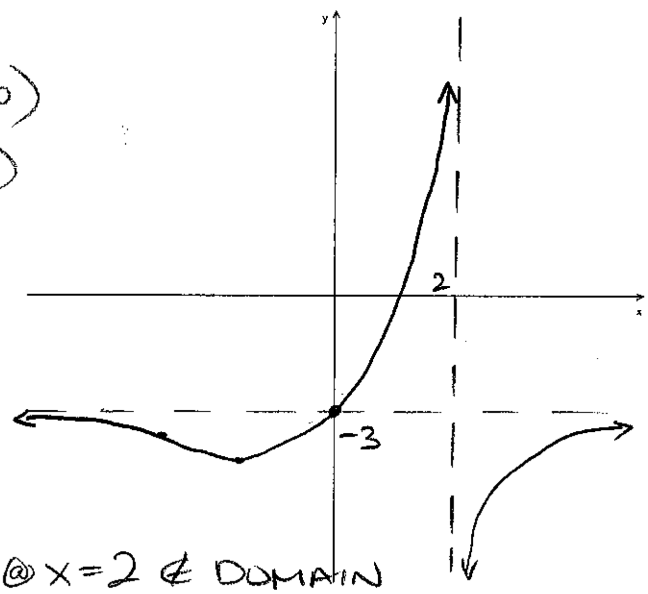
$\lim_{x\to 2^+} \left(\frac{x}{(2-x)^3} - 3\right) = -\infty \quad \left(\frac{2}{0^-} - 3 \to -\infty - 3 \to -\infty\right)$

$\lim_{x\to 2^-} \left(\frac{x}{(2-x)^3} - 3\right) = \infty \quad \left(\frac{2}{0^+} - 3 \to \infty - 3 \to \infty\right)$

$\lim_{x\to \pm\infty} \left(\frac{x}{(2-x)^3} - 3\right) = \left(\lim_{x\to\pm\infty} \frac{1}{-3(2-x)^2}\right) - 3$

$\frac{\infty}{\pm\infty} - 3$

$= 0 - 3 = -3$



$f'(x) = (2-x)^{-4}(2-x+3x)$

$= (2-x)^{-4}(2x+2)$

$= 2(2-x)^{-4}(x+1)$ UNDEFINED @ $x = 2 \notin$ DOMAIN

$= 0$ @ $x = -1$

$f''(x) = 6(2-x)^{-5}(2-x+2x)$

$= 6(2-x)^{-5}(2+x)$ UNDEFINED @ $x = 2 \notin$ DOMAIN

$= 0$ @ $x = -2$

f'	$-$		$-$		$+$		$+$
f''	$-$	IP	$+$	LOCAL MIN	$+$		$-$
		-2		-1		2	